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On inclination-flip homoclinic orbit associated to a saddle-node singularity

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Abstract. In this article, we study vector fields bifurcating through a saddle-node equilibrium with an unstable homiclinic orbit. Bifurcating diagrams for two-parameter perturbations of these vector fields are exhibited. It is proved that Smale's horseshoe dynamics, surrounding the bifurcating homoclinic orbit, exists for a large set of such perturbations.

1. Introduction

The complicated dynamical behavior arising from the existence of a homoclinic orbit have been much studied in the last thirty years or so. These investigations started with early works of Poincaré, about a hundred years ago, and afterwards Birkhoff. It is implicit in the works of Cartwright-Littlewood and Levinson, and its dynamic prototype as a horseshoe map is in the remarkable work of Smale [Sm]. The objetive of this paper is to analyze vector fields with a special homoclinic scenario consisting of an equilibrium point and a regular orbit converging to it in positive and negative time, a *homoclinic loop*. For vector fields (flows) on surfaces, homoclinic loops were considered in [AL]. Later references, specially for diffeomorphisms, are [NPT] and [PT].

To describe the main assumptions under which homoclinic loops for flows will be studied here, some preliminar notations and well known results are introduced. First of all, we restrict our study to C^r-vector fields in a three-dimensional closed manifold, where r is a positive large integer. Singularities of a given vector field X will be understood as those

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points σ such that $X(\sigma) = 0$. Throughout, a singularity σ of X is called *saddle-node* if the linear part $DX(\sigma)$ of X on σ has three real eigenvalues $\lambda_1, -\lambda_2, -\lambda_3$ satisfying $-\lambda_2 < -\lambda_3 = 0 < \lambda_1$ (see [So]). When σ is a saddle-node singularity of X, it follows from the invariant manifold theory (see [HPS]) that the set $W^s(\sigma)$ (resp. $W^u(\sigma)$), consisting of those points whose forward (resp. backward) X-orbit converges to σ, is a submanifold with boundary $W^{ss}(\sigma)$ (resp. $W^{uu}(\sigma)$). Both, $W^s(\sigma)$ and $W^u(\sigma)$, are usually called *center-stable* and *center-unstable* manifold respectively.

In this paper, we shall consider a homoclinic loop Γ such that its corresponding singularity σ is a saddle-node and such that $W^s(\sigma) \setminus W^{ss}(\sigma)$ and $W^u(\sigma) \setminus W^{uu}(\sigma)$ intersect nontransversally along Γ. This last assumption, often called *inclination-flip*, was studied in [KKO] in the case when the corresponding singular point is hyperbolic.

The motivations behind the study of saddle-node singularities exhibiting inclination-flip homoclinic loops come from several works. Homoclinic loops associated to saddle-node singularities in dimension two were studied in [AL]. In higher dimensions, a similar problem was treated in [Sh], but with the hypothesis of transversal intersection between the center-stable and center-unstable manifolds along the loop: it was proved that a unique (hyperbolic) periodic orbit is generated after saddle-node singularity disappears. In addition, such a periodic orbit is the unique nonwandering orbit in a fixed small neighborhood of the loop.

On the other hand, inclination-flip homoclinic loops associated to hyperbolic singularities were also considered. In fact, such loops were studied in [KKO] and in [HKK]. In the first work it is shown that periodic and homoclinic doubling bifurcations do occur under suitable conditions on the eigenvalues of the singularity. The second considered the problem of finding Smale's horseshoes in generic perturbations of an inclination-flip homoclinic orbit. It was proved that, under certain open conditions on the eigenvalues of the singularity, such a phenomena actually occurs at least in a thin region of the parameter space. By a thin region we mean a (two-dimensional) Lebesgue density zero set at the bifurcating parameter value. In [Ry] it was proved that geometric Lorenz attractors can bifurcate from a pair of inclination-flip homoclinic orbits associated to hyperbolic saddle-type singularities. Our main result is the following

Theorem A. *Let $\{X_\mu\}_{\mu \in R^2}$ be a generic two-parameter family of vector fields such that $X_{(0,0)}$ exhibits an inclination-flip homoclinic orbit Γ associated to a saddle-node singularity. For a small neighborhood U of Γ, the following two properties hold:*

1. the set of parameters μ for which the nonwandering set of X_μ (restricted to U) is either empty or a hyperbolic set has full two-dimensional Lebesgue density at $\mu = (0,0)$;

2. the set of parameters for which the nonwandering set of X_μ (restricted to U) is a suspended Smale's horseshoe has positive two-dimensional Lebesgue density at $\mu = (0,0)$.

The case of a saddle-node singularity exhibiting both inclination-flip and transversal homoclinic orbits is also considered. By a transversal homoclinic orbit we mean a homoclinic orbit which is not an inclination-flip. In [AS] it was proved that a suspended Smale's horseshoe of p-symbols arises from the disappearance of a saddle-node singularity with p-transversal homoclinic orbits, all of them belonging to the interior of the center manifolds of the singularity. This fact lead us to study situations which are obtained when it is unfolded a saddle-node singularity with p-transversal homoclinic orbits and just one inclination-flip homoclinic orbit. As a corollary of the proof of theorem A, we obtain

Theorem B. *Let $\{X_\mu\}_{\mu \in R^2}$ be a generic two-parameter family such that $X_{(0,0)}$ exhibits a saddle-node singularity having p-transversal homoclinic orbits and one inclination-flip homoclinic orbit, all of them lying in the interior of the corresponding center manifolds. Then, for a small neighborhood U of these homoclinic orbits, the set of parameters μ such that the nonwandering set of X_μ in U is either a hyperbolic set conjugated a subshift of p-symbols, a hyperbolic set conjugated to a subshift of $(p+2)$-symbols or two hyperbolic singularities has full two-dimensional Lebesgue density at $\mu = (0,0)$.*

We point out that there is some relation between the inclination-flip homoclinic loop considered here and the Lorenz attractor bifurcation process showed in [M]. In this reference, it was shown that a geometric Lorenz attractor can unfold into homoclinic tangencies through a codimension-one lamina formed by a special class of vector fields called saddle-node Lorenz attractors. It is easy to see, in such a codimension-one lamina, that there exist open regions having a dense set of vector fields all of them having a inclination-flip homoclinic orbit associated to a saddle-node singularity. The viewpoint we present here, concerning prevalence of certain features (positive density) in dynamic bifurcation, has been inspired by [PT].

This paper is organized in five sections. In section 2 a precise version of theorem A is presented (Theorem 2.2), together with some basic definitions. In section 3 we shall prove two previous propositions which are immediate consequences of the standard transversal theory. In section 4, an approximated bifurcation diagram is presented. This diagram yields full Lebesgue density regions where the nonwandering set, close to the homoclinic orbit, is either two hyperbolic singularities, empty or a topological horseshoe. Finally, in the last section, this region is refined in order to obtain hyperbolic horseshoes.

2. Inclination-flip saddle-node homoclinic orbit

Here we consider a three-dimensional vector field X_0 which has a singularity at $(x, y, z) = (0, 0, 0)$. We assume that it is a saddle-node singularity, i.e. $DX_0(0, 0, 0)$ has three real eigenvalues $-\lambda_2 < -\lambda_3 = 0 < \lambda_1$. It follows from the invariant manifold theory (see [HPS]) that there exist the next invariant manifolds associated to $\mathbf{0}=(0, 0, 0)$: $W^s(\mathbf{0})$, $W^{cs}(\mathbf{0})$, $W^u(\mathbf{0})$, $W^{cu}(\mathbf{0})$ and $W^c(\mathbf{0})$ which are tangent to the set of eigenvectors associated to the corresponding set of eigenvalues given by $\{-\lambda_2\}$, $\{-\lambda_2, 0\}$, $\{\lambda_1\}$, $\{\lambda_1, 0\}$ and $\{0\}$. They are called the stable, center-stable, unstable, center-unstable and the center manifolds respect.. In addition, these manifolds are as smooth as the initial flow X_0.

We also assume the existence of a homoclinic orbit associated to the origin $\mathbf{0}$, that is, there exists a solution of X_0, namely $h(t)$, such that it goes to zero when $t \to \pm\infty$. Let us set $\Gamma = Cl(\{h(t) : t \in R\})$. We assume the following hypotheses respect to the homoclinic orbit Γ: (N-P) Γ is in $(W^{cu}(\mathbf{0}) \cap W^{cs}(\mathbf{0})) \setminus (W^u(\mathbf{0}) \cap W^s(\mathbf{0}))$.

(Q-T) $W^{cu}(\mathbf{0})$ and $W^{cs}(\mathbf{0})$ have quadratic tangency along Γ.

A homoclinic orbit as Γ is called *inclination-flip saddle-node homoclinic orbit.* Let us define $\mathcal{W}$ as the set of three-dimensional vector fields which have a inclination-flip saddle-node homoclinic orbit.

Proposition 2.1. *Let $\mathcal{W}$ be as above. Then $\mathcal{W}$ is a codimension two submanifold in the space of three-dimensional vector fields.*

Based on this proposition, we deal with two-parameters families which are transversal to $\mathcal{W}$. To start, we consider a small neighborhood U of its corresponding inclination-flip saddle-node homoclinic orbit and we say that a parameter μ is *hyperbolic relative to U* iff the nonwandering set of the corresponding vector field, in U, is either a empty set or a hyperbolic set. Also we say that μ is a *suspended horseshoe parameter relative to U* iff such a nonwandering set is a suspended Smale's horseshoe. Theorem A can now be stated as follows:

Theorem 2.2. *Let $\{X_\mu\}_{\mu \in R^2}$ be a two-parameter family of three-dimensional vector fields which is transverse to $\mathcal{W}$ at $\mu = (0,0)$ and consider a small neighborhood U of the corresponding inclination-flip saddle-node homoclinic orbit. Then, the set of hyperbolic parameters relative to U has Lebesgue density one at $\mu = (0,0)$. Moreover the set of suspended horseshoe parameters relative to U has positive Lebesgue density at the same parameter value.*

In the proof of this theorem we will see that parameters corresponding to vector field which has empty nonwandering set in U have also positive Lebesgue density at $\mu = (0,0)$. In addition, it will be clear that $\mu = (0,0)$ is also accumulated by positive Lebesgue measure sets of parameters which correspond to vector field with a Henon-like strange attractors (see [MV] and [PT]). A different unfolding of the horseshoe (see [PT]) is explained in the sequel. Briefly, the unfolding consists of the disappearance of the horseshoe through a fixed parabolic curve.

3. Basic propositions

Suppose that $\{X_\mu\}_{\mu \in R^2}$ is a two-parameter family of three-dimensional vector fields such that $X_0 \in \mathcal{W}$. Our goal here is to give some basic propositions and also conditions to test when such a family is transverse to $\mathcal{W}$ at $\mu = (0,0)$.

The first main assumption is related to the behavior of X_μ close to the saddle-node singularity attached at $\mu = \mathbf{0}$ which we suppose that is the origin $\mathbf{0} = (0,0,0)$. We assume that the following expression for the given vector field holds.

$$X_\mu(x,y,z) = (\lambda_1 + f(z,\mu))\partial_x + (-\lambda_2 + g(z,\mu))\partial_y + H(z,\mu)\partial_z$$

with

$$\mu = (\mu_1, \mu_2) \in R^2 \ ; \ f(0,0,0) = g(0,0,0) = 0 \ ; \ \lambda_i > 0$$

and $H(z,\mu)$ being a saddle-node arc in the following sense: $H(0,0,0) = H_z(0,0,0) = H_{\mu_2}(0,0,0) = 0$; $H_{\mu_1}(0,0,0) = -b < 0$ and $H_{zz}(0,0,0) = -2.a < 0$.

This is a generic condition (see [T]) and can be dropped using the methods given by [D]. That is assumed for the sake of simplicity. We also assume that the following global properties hold for the initial field X_0.

$$\begin{cases} W^{cu}_{loc}(\mathbf{0}) \cap W^{cs}(\mathbf{0}) \subset \{ \ x = y = 0 \ \} \\ W^{cs}_{loc}(\mathbf{0}) \cap W^{cu}(\mathbf{0}) \subset \{ \ x = y = 0 \ ; z > 0 \ \} \end{cases}$$

where $W^i_{loc}(\mathbf{0})$ $(i = cu, cs)$ are the local center-unstable and center-stable manifolds (resp.) associated to the saddle-node $\mathbf{0}$. Let us consider two transversal sections: $\Sigma_0 = \{ \ |x|, |y| \leq 1 \ , \ z = 1 \ \}$ and $\Sigma_1 = \{ \ |x|, |y| \leq 1 \ , \ z = -1 \ \}$.

Also we consider the two natural coordinate systems (x_0, y_0) and (x,y) in Σ_0 and Σ_1 respec.. Then a successive flow-defined map G_μ is defined for μ close to $\mathbf{0}$:

$$G_\mu(x,y) = (g^1(x,y,\mu), g^2(x,y,\mu))$$

from Σ_0 to Σ_1 and it has the following properties:

$$\begin{cases} G_{(0,0)}(0,0) = 0 \\ g_x^1(0,0,0,0) = 0 \ ; \ g_{xx}^1(0,0,0,0) \neq 0. \end{cases}$$

because of $X_{(0,0)} \in \mathcal{W}$.

Proposition 3.1. *A two-parameter family $\{X_\mu\}_{\mu \in R^2}$, as the above considered, is transverse to $\mathcal{W}$ at $\mu = (0,0)$ iff $g_{\mu_2}(0,0,0,0) \neq 0$.*

Proof. Using $g_{xx}^1(0,0,0,0), H_{zz}(0,0,0) \neq 0$ we obtain two functions $\mu \to x(\mu)$ and $\mu \to z(\mu)$ such that

$$g_x^1(x(\mu), 0, \mu) = H_z(z(\mu), \mu) = 0 \quad x(0,0) = z(0,0) = 0.$$

Now we define $\psi(\mu) = (g^1(x(\mu), 0, \mu), H(z(\mu), \mu))$. Then $X_\mu \in \mathcal{W}$ iff $\psi(\mu) = (0,0)$. Therefore, the family is transversal to $\mathcal{W}$ at $\mu = (0,0)$ iff $D\psi(0,0)$ is not a singular matrix. But this last statement is true iff $g_{\mu_2}^1(0,0,0,0) \neq 0$. The proof is complete.

Next proposition give us a first aproximating bifurcation diagram for a two-parameter family transverse to $\mathcal{W}$.

Proposition 3.2. *Suppose that $\{X_\mu\}_{\mu \in R^2}$ is a two-parameter family as above-mentioned and it is transversal to $\mathcal{W}$ at $\mu = (0,0)$. Then there exist a smooth change of parameters $v = v(\mu)$ and a small neighborhood$U$ of the homoclinic loop at $\mu = (0,0)$ such that:*

a. *for $v_1 < 0$, the nonwandering set of X_v relative to U consists in two hyperbolic singularities and therefore v is a hyperbolic parameter relative to U;*

b. *for $v_1 = 0$ the nonwandering set of X_v in U is a unique singularity of saddle-node type;*

c. *for $v_2 = 0$, there is a quadratic tangency between the invariant manifolds $W^{cs}(v)$ and $W^{cu}(v)$, which are the analytic continuation of $W^{cs}(\mathbf{0})$ and $W^{cu}(\mathbf{0})$ respectivelly.*

Proof. This proposition follows by standard arguments about generics unfolding of a saddle-node singularity: Let us consider the functions $x(\mu)$ and $z(\mu)$ of proposition 3.1. Then, the equation $H(z(\mu), \mu) = 0$ has a solution function $\mu_2 \to \hat{\mu}_1(\mu_2)$ for which $\hat{\mu}_1'(0) = 0$. Also the equation

$g^1(x(\mu), 0, \mu) = 0$ give us a solution function $\mu_1 \to \hat{\mu}_2(\mu_1)$ which has finite derivative at the origin. Let us define the following change of parameters:

$$\begin{cases} v_1 = \mu_1 - \hat{\mu}_1(\mu_2) \\ v_2 = \mu_2 - \hat{\mu}_2(\mu_1) \end{cases}$$

This completes the proof of proposition 3.2.

Remark. If we still denote by H and G the expression of the corresponding maps with respect to the new parameters, then $g^1_{v_1}(0,0,0,0) = 0$. Moreover, the following expansion holds:

$$H(z, v) = g(v_2).v_1 + B(v).(z - h(v))^2 + O(|z - h(v)|^3 + |v_1|^3)$$

where g, b and h are smooth functions with $g(0,0) < 0$, $B(0,0) < 0$ and $h(0,0) = 0$. Indeed, if we expand H around $z(v)$, it follows that:

$$\begin{aligned} H(z, v) =& H(z(v), v) + H_z(z(v), v).(z - z(v)) + \\ &+ (1/2).H_{zz}(z(v), v).(z - z(v))^2 + O(|z - z(v)|^3) \end{aligned}$$

then we set $g(v_2) = \varphi_{v_1}(0, v_2)$ $\big(\varphi(v) = H(z(v), v)\big)$, $B(v) = (1/2).H_{zz}(z(v), v)$ and $h(v) = z(v)$. Throughout we assume that $v = \mu$, for the sake of simplicity.

4. Topological horseshoe region

The approximating bifurcation diagram given in proposition 3.2 says that μ is a hyperbolic parameter while $\mu_1 < 0$. Here we consider the case $\mu_1 > 0$. There is a successive flow-defined map Π_μ from Σ_0 to Σ_1 when $\mu_1 > 0$. It give us a Poincare map $F_\mu = G_\mu \circ \Pi_\mu$. The nonwandering set of X_μ in a small neighborhood U of the homoclinic loop at $\mu = (0,0)$ is the suspension of the nonwandering set of F_μ in Σ_0, for $\mu_1 > 0$. Π_μ has the following form, for $\mu_1 > 0$.

$$\Pi_\mu(x_0, y_0) = (\sigma(\mu).x_0, \lambda(\mu).y_0)$$

where

$$\sigma(\mu) = e^{[\lambda_1.T(\mu) + \int_0^{T(\mu)} f(z,\mu)d\theta]},$$

$$\lambda(\mu) = e^{[-\lambda_2.T(\mu) + \int_0^{T(\mu)} g(z,\mu)d\theta]},$$

$$T(\mu) = -\int_{-1}^{1} \frac{ds}{H(s, \mu)}.$$

We are going to assume that $g^1_{xx}(0,0,0,0) > 0$, $g^1_{\mu_2}(0,0,0,0) < 0$. The remaining cases can be handled in the same way.

The main result of this section is the following

Proposition 4.1. *Let $\{X_\mu\}_{\mu \in R^2}$ a two-parameter family of vector field which are transverse to $\mathcal{W}$ at $\mu = (0,0)$ and as the one given before (see §3). Then there exist two functions $K^-, K^+ : \{ \mu_1 \geq 0 \} \to R$ such that the following hold:*

b. *$K^{\pm}(0) = 0$, $K^+(x) > 0$ and $K^-(x) < 0$. Also $(K^{\pm})'(0) = 0$;*

a. *if $\mu = (\mu_1, \mu_2)$ is close to zero, $\mu_1 > 0$ and $0 < \mu_2 < K^-(\mu_1)$ then the nonwandering set of X_μ, in some fixed small neighborhood U of the homoclinic loop, is empty;*

c. *if $\mu_1 > 0$ and $\mu_2 > K^+(\mu_1)$, then the nonwandering set of X_μ in U is a suspended topological horseshoe.*

Before the proof of this proposition, we consider the map Π_μ associated to the singularity. Let U be a small neighborhood of the homoclinic loop at $\mu = 0$ and suppose that it contains the cube $[-1,1]^3$. Let $A(\mu)$ be the preimage of $[-1,1]^2 \times \{-1\}$ through the above considered map and, at the same time, let $B(\mu)$ be the image of $A(\mu)$ under Π_μ. Then we have the following expressions,

$$\begin{cases} A(\mu) = [-\sigma^{-1}(\mu), \sigma^{-1}(\mu)] \times [-1,1] \times \{1\} \\ B(\mu) = [-1,1] \times [-\lambda(\mu), \lambda(\mu)] \times \{-1\} \end{cases}$$

Define $x^+(\mu)$ and $x^-(\mu)$ by the equations

$$g^1_x(x^{\pm}(\mu), \pm\lambda(\mu), \mu) = 0, \quad \mu \in \{ \mu_1 > 0 \}$$

Moreover, we define $c^+(\mu)$ and $c^-(\mu)$ by

$$c^{\pm}(\mu) = g^1(x^{\pm}(\mu), \pm\lambda(\mu), \mu), \quad \mu \in \{ \mu_1 > 0 \}$$

These last two functions are the critical values of the corresponding maps $x \to g^1(x, \pm\lambda(\mu), \mu)$. We have

Lemma 4.1. *Suppose that $\mu \in \{ \mu_1 > 0 \}$, $\mu_2 > 0$ and*

$$max\{ c^+(\mu),\ c^-(\mu)\} < -\sigma^{-1}(\mu)$$

(this is called the topological horseshoe condition, T.H.C.), then the nonwandering set of the corresponding vector field in U is a topologi-

cal horseshoe. On the other hand, assume that $\mu \in \{ \mu_1 < 0 \}$, $\mu_2 < 0$, *and*

$$min\{ c^+(\mu),\ c^-(\mu)\} > \sigma^{-1}(\mu)$$

(this is called the empty condition, E.C.), then the nonwandering set of corresponding vector field is a empty set.

This lemma already implies that topological horseshoe and empty nonwandering set appear in parameter regions given by inequalities as the following one.

$$c^{\pm}(\mu) + (k+1).\sigma^{-1}(\mu) + l.\lambda(\mu) \neq 0$$

for some positive fixed constants k and l.

Lemma 4.2. *Let* $\varphi_{k,l}(\mu)$ *be the map defined by*

$$\varphi_{k,l}(\mu) = \varphi(\mu) = c^s(\mu) + (k+1).\sigma^{-1}(\mu) + l.\lambda(\mu)$$

with $s = +,-$ *and* $\mu \in \{ \mu_1 > 0 \}$. *Then* φ *has a* C^1*-extension in a whole neighborhood of* $\mu = (0,0)$. *If this extension is still denoted by* φ, *then* $\varphi_{\mu_1}(0,0) = 0$ *and* $\varphi_{\mu_2}(0,0) \neq 0$.

Proof. Here we consider the case $s = +$. By doing elementary computations we obtain that

$$\varphi_{\mu_i}(\mu) = (g^1_y(x^+(\mu), \lambda(\mu), \mu) + l).\lambda_{\mu_i}(\mu) + g^1_{\mu_i}(x^+(\mu), \lambda(\mu), \mu) - (k+1).\frac{\sigma_{\mu_i}(\mu)}{\sigma^2(\mu)}$$

if $\mu \in \{ \mu_1 > 0 \}$ (i=1,2) but

$$\lambda_{\mu_i}(\mu) = \lambda(\mu).\Big\{ (-\lambda_2 + g(-1,\mu)).T_{\mu_i}(\mu) + \\ + \int_0^{T(\mu)} [g_z(z(\theta,\mu),\mu).z_{\mu_i}(\theta,\mu) + g_{\mu_i}(z(\theta,\mu),\mu)]d\theta \Big\}$$

and

$$\sigma_{\mu_i}(\mu) = \sigma(\mu).\Big\{ (\lambda_1 + f(-1,\mu)).T_{\mu_i}(\mu) + \\ + \int_0^{T(\mu)} [f_z(z(\theta,\mu),\mu).z_{\mu_i}(\theta,\mu) + f_{\mu_i}(z(\theta,\mu),\mu)]d\theta \Big\}$$

Claims. For any real number b close to zero the following holds:

a. $|\sigma^{-1}(\mu).T(\mu)|$ and $|\lambda(\mu).T(\mu)|$ go to zero when $\mu \in \{ \mu_1 > 0 \}$ goes to $(0,b)$.

b. $|\sigma^{-1}(\mu).T_{\mu_i}(\mu)|$ and $|\lambda(\mu).T_{\mu_i}(\mu)|$ go to zero when $\mu \in \{ \mu_1 > 0 \}$ goes to $(0, b)$.

If both claims hold the required C^1-extension exists by setting $\varphi(\mu) = g^1(x(\mu), 0, \mu)$ (recall proposition 3.1) for $\mu \in \{ \mu_1 \leq 0 \}$. In order to prove both claims, we adapt arguments given in [DRV] as follows:

First we observe that for every $\delta > 0$ and every nonnegative integer i:

$$\limsup_{\mu \to (0,b)} [-g(\mu_2).\mu_1]^{(i+1/2)}. \int_{-1}^{1} \frac{ds}{X^{(i+1)}} = \limsup_{\mu \to (0,b)} [-g(\mu_2).\mu_1]^{(i+1/2)}. \int_{-\delta}^{\delta} \frac{ds}{X^{(i+1)}}$$

(recall remark after proposition 3.2 §2) where X is either $H(s, \mu)$ or $\tilde{H}(s, \mu) = -g(\mu).\mu_1 - B(\mu).(z - h(\mu))^2$. Also, we have that for any $\epsilon > 0$ there exist $\delta > 0$ and $\Delta > 0$ such that

$$1 - \epsilon \leq \frac{-H(s, \mu)}{\tilde{H}(s, \mu)} \leq 1 + \epsilon$$

if $(s, \mu) \in [-\delta, \delta] \times \{ |\mu| < \Delta \}$, because of the remark after proposition 3.2 applies. Then

$$\limsup_{\mu \to (0,b)} [-g(\mu_2).\mu_1]^{(i+1/2)} \int_{-1}^{1} \frac{ds}{[-H(s, \mu)]^{(i+1)}} \leq$$
$$\leq (1 - \epsilon)^{-(i+1)}. \lim_{\mu \to (0,b)} [-g(\mu_2).\mu_1]^{(i+1/2)} \int_{-1}^{1} \frac{ds}{[\tilde{H}(s, \mu)]^{(i+1)}}$$

Moreover,

$$\liminf_{\mu \to (0,b)} [-g(\mu_2).\mu_1]^{(i+1/2)} \int_{-1}^{1} \frac{ds}{[-H(s, \mu)]^{(i+1)}} \leq$$
$$\leq (1 + \epsilon)^{-(i+1)} \lim_{\mu \to (0,b)} [-g(\mu_2).\mu_1]^{(i+1/2)} \int_{-1}^{1} \frac{ds}{[\tilde{H}(s, \mu)]^{(i+1)}}$$

and therefore

$$\lim_{\mu \to (0,b)} [-g(\mu_2).\mu_1]^{(i+1/2)} \int_{-1}^{1} \frac{ds}{[-H(s, \mu)]^{(i+1)}} =$$
$$= \lim_{\mu \to (0,b)} [-g(\mu_2).\mu_1]^{(i+1/2)} \int_{-1}^{1} \frac{ds}{[\tilde{H}(s, \mu)]^{(i+1)}}$$

provided the second limit exists, since $\epsilon > 0$ is arbitrary.

It can be proved, by a direct computation, that this last limit exists and it is a positive number $C(i)$ for all i. When $i=0$ we get

$$|T(\mu)| \approx e^{C(0)/\sqrt{-g(\mu_2).\mu_1}}$$

and then $|T(\mu)|$ goes to infinity if μ goes to $(0,b)$, hence claim (a) follows. In the case $i=1$ we have

$$\int_{-1}^{1} \frac{ds}{[H(s,\mu)]^2} \approx \frac{C(1)}{[-g(\mu_2).\mu_1]^{3/2}}$$

then (b) follows.

Now, proposition 4.1 follows by applying lemma 4.2 with $l=0$, $k=0$, $s=+$ for T.H.C. and $s=-$ for E.C..

5. Hyperbolic region: proof of theorem 2.2

In order to prove theorem 2.2 we study the case $\mu_1>0$ in details. More precisely, we search the topological horseshoe region (given in §4) to get conditions which leads hyperbolic horseshoes.

In fact, not all of the topological horseshoe appearing in the Topological horseshoe region are hyperbolic ones, but it will be proved that hyperbolicity do occur in most of the cases.

Let us consider the region given by

$$g^1(x^+(\mu),\lambda(\mu),\mu)+k.\sigma^{-1}(\mu)+l.\lambda(\mu) < -\sigma^{-1}(\mu) \;\; ; \;\; \mu \in \{\ \mu_1>0\ \}$$

where k,l are positive constants. Then, by using lemma 4.2, it follows that if we define the function $\tilde{\varphi}(\mu)$ as

$$\tilde{\varphi}(\mu) = g^1(x^+(\mu),\lambda(\mu),\mu)+(k+1).\sigma^{-1}(\mu)+l.\lambda(\mu)$$

then the equation $\tilde{\varphi}(\mu)=0$ is solved by a C^1-function $\mu_2=\widehat{\mu}_2(\mu_1)>K^+(\mu_1)$ such that $\widehat{\mu}_2(0)=\widehat{\mu}'(0)=0$. The main result of this section is the following

Proposition 5.1. *There exist l and k large such that if μ is a parameter value in $\{\ \mu_1>0\ \}$ and the following condition holds.*

$$g^1(x^+(\mu),\lambda(\mu),\mu) < -(k+1).\sigma^{-1}(\mu)-l.\lambda(\mu)$$

(this is called the hyperbolic condition, H.C.), then μ is a suspended horseshoe parameter relative to U.

Before the proof of this proposition, we give some previous lemmas.

Lemma 5.2. *For every* $\epsilon > 0$, *there exists a positive number* Δ *such that the following holds: there exist two* C^1*-functions*

$$x_+, x_- : [-\lambda(\mu), \lambda(\mu)] \times [-\sigma^{-1}(\mu), \sigma^{-1}(\mu)] \times \{ \mu : |\mu| < \Delta \text{ } H.C. \text{ holds} \} \longrightarrow R$$

for which

$$g^1(x_\pm(y,\xi,\mu), y, \mu) = \xi$$

and

$$K.\frac{|x_\pm(y,\xi,\mu) - x^+(\mu)|}{\sqrt{\xi - c^+(\mu) - F.(y - \lambda(\mu))}} \in [1-\epsilon, 1+\epsilon]$$

for some positive fixed constant F *and* K.

Proof. The existence of both x_+ and x_- follows by the quadratic nature of g^1 respect to x. Now, we have

$$\begin{aligned} g^1(x,y,\mu) =& c^+(\mu) + \left\{ d(\mu) + \frac{\tilde{g}^1(x,\mu)}{(x - x^+(\mu))^2} \right\}.(x - x^+(\mu))^2 + \\ &+ \left\{ f^*(x,\mu) + \frac{R(x,y,\mu)}{(y - \lambda(\mu))} \right\}.(y - \lambda(\mu)) \end{aligned}$$

for some appropriates functions $\tilde{g}^1, f^*, R$ and d, such that

$$\tilde{g}^1(0,\mu) = \tilde{g}^1_x(0,\mu) = \tilde{g}^1_{xx}(0,\mu) = R(x,0,\mu) = R_y(x,0,\mu) = 0$$

for every μ and $d(0,0), f^*(0,0,0) > 0$. Thus the following identity holds.

$$x_\pm(y,\xi,\mu) - x^+(\mu) = \pm\sqrt{\frac{\xi - c^+(\mu) - [f^*(x,\mu) + \frac{R(x_\pm, y, \mu)}{(y-\lambda(\mu))}].(y - \lambda(\mu))}{d(\mu) + \tilde{g}^1(x_\pm,\mu).x_\pm^{-2}}}$$

hence we are done iff μ is small enough.

Lemma 5.3. *For every* $\epsilon > 0$ *there exists* $\Delta > 0$ *such that*

$$\frac{|y - \lambda(\mu)|}{|x_+(y,\xi,\mu) - x^+(\mu)|} < \epsilon$$

if $(y,\xi) \in [-\lambda(\mu), \lambda(\mu)] \times [-\sigma^{-1}(\mu), \sigma^{-1}(\mu)]$, $\mu \in \{ \mu_1 > 0 \}$, $|\mu| < \Delta$ *and H.C. holds with* l *large enough.*

Proof. Using lemma 5.2 one has

$$\frac{|y - \lambda(\mu)|}{|x_+ - x^+(\mu)|} \leq K.\frac{2\lambda(\mu)}{\sqrt{\xi - c^+(\mu) - F.(y - \lambda(\mu))}}$$

now

$$\xi - c^+(\mu) - F.(y - \lambda(\mu)) > k\sigma^{-1}(\mu) + (l - 2F)\lambda(\mu)$$

because H.C. holds. Then we are done if μ is small enough and l is large enough.

Now, let us define the following cone family

$$C_\epsilon^+(y, \xi) = \{ (u, v) \ : \ |\frac{u}{v}| < \epsilon \}$$

this induces a cone field in $G_\mu(B(\mu)) \cap A(\mu)$ (recall §3 and §4).

Lemma 5.4. *Suppose that $\mu \in \{ \mu_1 > 0 \}$ is a parameter value for which H.C. holds. Then the angle between the horizontal lines and the vector $(g_x^1, g_x^2)(x_+(y, \xi, \mu), y, \mu)$ is less than*

$$\frac{K'}{\sqrt{k\sigma^{-1}(\mu) + l'.\lambda(\mu)}}$$

for some constants $K', l' > 0$ where l' is as large as l.

Proof. First we look at the following quotient:

$$q(\mu) = \left| \frac{g_x^2(x_+(y, \xi, \mu), y, \mu)}{g_x^1(x_+(y, \xi, \mu), y, \mu)} \right|$$

it follows that

$$q(\mu) = |x_+ - x^+|^{-1}. \left| \frac{c_1}{c_2 + c_3.[(y - \lambda(\mu))/(x_+ - x^+)]} \right|$$

for some functions c_i such that c_1, c_2 are positives at $\mu = (0, 0)$. Then, using lemma 5.3, we have

$$|q(\mu)| \leq \frac{c_4}{|x_+ - x^+|} \leq \frac{K'}{\sqrt{k\sigma^{-1}(\mu) + l'\lambda(\mu)}}$$

Back to the proof of proposition 5.1, let us define

$$\alpha(\mu) = \frac{K'}{\sqrt{k\sigma^{-1}(\mu) + l'\lambda(\mu)}}.$$

Then, the cone field given by $\{ C_{2\alpha(\mu)}(y, \xi) \}$ is well defined and is carried by Π_μ into the cone field $\tilde{C}_{\beta(\mu)}(x, y) = \{ (u, v) \ : \ |u/v| < \beta(\mu) \}$, where $\beta(\mu) = c.\sigma^{-1}(\mu).\lambda(\mu).\alpha(\mu)$ and thus $D(G_\mu \circ \Pi_\mu)(C_{\alpha(\mu)}(y, \xi, \mu)) \subset \text{int}(C_{\alpha(\mu)}(y', \xi'))$ for some y', ξ' because of $(k\sigma^{-1}(\mu) + l'\lambda(\mu))^{-1}.\sigma^{-1}(\mu).\lambda(\mu)$ goes to zero if μ goes to zero. Finally, by doing straightforward computations, expansivity of both $D(G_\mu \circ \Pi_\mu)$ and $D(G_\mu \circ \Pi_\mu)^{-1}$ in $C_{2\alpha(\mu)}(y,\xi)$ and $R^2 \backslash C_{2\alpha(\mu)}(y,\xi)$ respect. follow. Then the hyperbolicity follows as in [PT]. This finishes the proof of proposition 5.1.

Now, the proof of theorem 2.2 is completed using proposition 5.1 and proposition 4.1.

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